# Topology Preserving Simplification of Meshes with Embedded Structures

A Project Report

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# Abstract

Several visualization applications require simplification of high resolution meshes for faster processing. Many of these meshes contain interesting substructures, called embedded structures, within the mesh. There are applications that require the topology of the embedded structure as well as the mesh to be preserved during the simplification process. Such a simplification technique that uses edge contractions has been recently developed and shown to work on different datasets. This technique constructs what is known as an extended complex and contracts edges that satisfy link conditions on the extended complex. In this project, we prove mathematically that such edge contractions preserve the topology of the mesh and the embedded structures. We allow embedded structures to be on the boundary of the mesh and propose modifications to the existing algorithm to handle such cases. We also show that evaluation of link conditions are almost always necessary in ensuring that topology is preserved.

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# Chapter 1

# Introduction

Several modelling and simulation applications produce complex meshes at a very high level of detail. In order to speed up the subsequent processing, the meshes have to be simplified to generate a lower resolution approximation of the original mesh. A popular technique to simplify the mesh is to iteratively contract edges of the mesh. Many applications require that the topology of the mesh remain unchanged after the simplification process. In the context of edge contractions, topology preservation can be ensured by evaluating for each edge a set of conditions called link conditions [9] and allowing only those edges that satisfy the link conditions to be contracted.

Many meshes contain interesting substructures of lower dimensions embedded within the mesh. The topology of such embedded structures is often important. For instance, in a mesh that contains two distinct regions, the boundary separating the two regions could be an embedded structure whose topology needs to be preserved while simplifying the mesh.

Vivodtzev et al. [1, 2] have proposed a simplification technique that preserves topology of the mesh and the embedded structures. They transform the original mesh to an extended complex by attaching simplices from a dummy vertex to simplices of the embedded structure. Link conditions are evaluated on the extended complex and only those edges that satisfy the link conditions are contracted to simplify the mesh. They use this technique to simplify meshes with embedded structures and demonstrate using different datasets that the simplification process preserves the topology of the mesh and the embedded structures. However, the theoretical correctness of this technique is yet to be established.

The main contributions of this project are:

- We give a mathematical proof to show that link conditions evaluated in the extended complex of a 3D mesh preserves the topology of the mesh and embedded structure.

- Our proof for 3D meshes is generic and extends to 2D meshes also. The earlier proof given by Vivodtzev et al. [2] is incomplete as their analysis ignores simplices that are added from the dummy vertex.

- We extend the simplification algorithm to handle embedded structures that lie on the boundary. This was a limitation of the previous algorithm.
- We demonstrate the usefulness of the simplification in applications like isosurface preservation and molecular surface preservation.
- We show that evaluation of link conditions on the extended complex is necessary for topology preservation of a restricted class of 2D meshes with 1D embedded structures.

Our implementation of the simplification algorithm for tetrahedral meshes uses existing ideas based on the Quadric Error Metric (QEM) to improve the quality of mesh elements and to approximate the scalar field defined on the mesh. Further, the algorithm also preserves the geometry of the embedded structure. Evaluation of the link conditions requires the computation of the order of a simplex. This computation is non-trivial in the context of embedded structures. We describe an explicit characterization of simplices that leads to an algorithm for computing their order.

The rest of the report is organised as follows. Chapter 2 describes related work. Chapter 3 gives the definition of the terms used in this report. Chapter 4 describes the proofs in detail. Chapter 5 discusses implementation of the simplification algorithm. Chapter 6 lists the applications of the algorithm. Chapter 7 discusses the results of our implementation. Chapter 8 concludes the report.

# Chapter 2

# Related Work

## 2.1 Mesh Simplification

Mesh simplification is an area of active research in the area of scientific visualization. Surveys on different mesh simplification techniques can be found in [3, 4, 5]. Edge contractions are extensively used for mesh simplification and several algorithms exist that differ in the manner in which edges are chosen for contraction. A notable algorithm is the quadric error based algorithm of Garland et al. [6], which produces high quality approximations and is very efficient.

## 2.2 Topology Preservation and Controlled Simplification

Dey et al. [9] showed that edges that satisfy a set of conditions, called link conditions, can be contracted without causing topology violations. These are local conditions evaluated in the neighbourhood of the edge. However, link conditions do not distinguish embedded structures from the rest of the mesh and hence may not ensure topology preservation of embedded structures.

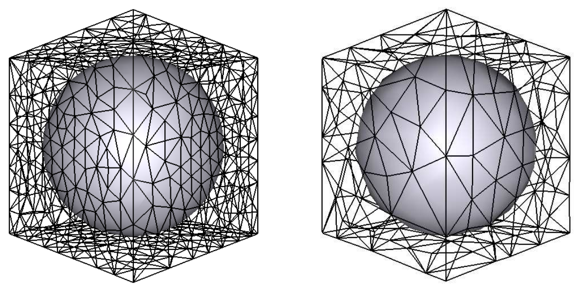
Early work on minimum and minimal triangulations studied the smallest possible mesh that can be reached without violating the topology [15, 16].

Different from topology preserving simplification, controlled topology simplification helps to remove topological noises like small holes, while retaining important topological characteristics of the mesh. Reeb graphs and Morse-Smale complexes are extensively used for controlled topological simplification [10, 11, 12].

## 2.3 Attribute Preservation

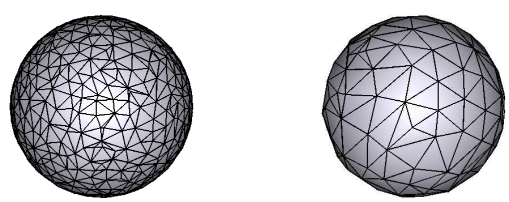
Various attributes like material colour, scalar field etc. are often available as data at each vertex of the mesh. These attributes also need to be preserved during the simplification process. Quadric error metric based simplification methods can be easily extended for attribute preservation [7, 8]. Cignoni et al. compare various simplification techniques to approximate the scalar field of a tetrahedral mesh [3].

## 2.4 Substructure Preservation



(a) (b)

Figure 2.1: (a) Original and (b) simplified volume containing embedded structure shown in grey.



(a) (b)

Figure 2.2: (a) Embedded structure before and (b) after simplification.

Mesh simplification algorithms should preserve important substructures of the mesh. We assume that the substructure is specified by the user or is available as the output of prior analysis of the mesh. Figure 2.1(a), 2.2(a) show the surface of a grey spherical ball embedded inside a cube. The mesh and the embedded structure after simplification is shown in Figure 2.1(b), 2.2(b).

By transforming the input mesh to an extended complex, Vivodtzev et al. [1, 2] encode the topology of the substructures in a new mesh and ensure that their topology is preserved during simplification.

They sketch a proof for topology preservation in the case of 2D meshes. However, this proof has a major gap since their analysis does not consider the new simplices in the extended complex added from the dummy vertex. Moreover, the proof does not extend to the case of 3D meshes.

## 2.5 Scalability

When the size of the mesh becomes huge, out-of-core algorithms are required to process the mesh. These techniques design a mesh representation scheme and a simplification algorithm that accesses the mesh in a spatially coherent manner [13, 17]. Since link conditions are evaluated in the neighbourhood of an edge, they can also be implemented out-of-core for large meshes.

# Chapter 3

# Definitions

A *k-simplex*, $\eta$ is the convex hull of $k+1 \geq 1$ affinely independent points. We denote a simplex with vertices $v_1, v_2, \ldots, v_k$ as $<v_1, v_2, \ldots, v_k>$. Its *dimension* is $dim(\eta) = k$. A *face* $\tau$ of $\eta$ is the simplex defined by a non-empty subset of the $k+1$ points and $\tau$ is *proper* if the subset is proper. We say $\tau \leq \eta$ and call $\eta$ a *coface* of $\tau$. The *interior* of a simplex, *int* $\tau$ is the set of points contained in $\tau$ but not in any proper face of $\tau$. A *simplicial complex* $K$ is a collection of simplices such that:

i. If $\eta \in K$ then all faces of $\eta$ are also in $K$

ii. If $\eta, \tau \in K$ then $\eta \cap \tau$ is empty or a face of $\eta, \tau$.

The dimension of $K$, $dim(K)$, is the largest dimension of simplices in $K$. The *underlying space* of $K$, $|K|$ is the union of simplex interiors in $K$. A simplex in $K$ is *principal* if it has no coface in $K$ other than itself. A *subdivision* of a simplicial complex $K$ is a simplicial complex $L$ such that $|L| = |K|$ and each simplex in $L$ belongs to one of the simplices in $K$. Two simplicial complexes $K$ and $L$ are said to be combinatorially equivalent, $K \simeq L$, if they have isomorphic subdivisions.

For $L \subseteq K$, the *closure* of $L$, denoted by $\overline{L}$ is the smallest subcomplex that contains $L$. The *star* of $L$ in $K$, denoted by $St(L; K)$, is the set of cofaces of the simplices in $L$. The *link* of $L$ in $K$, denoted by $Lk(L; K)$, is the set of all faces of simplices in the closure of star that are disjoint from simplices in $L$.

$$\begin{aligned} \overline{L} &= \{\tau \in K | \tau \leq \eta \in L\} \\ St(L; K) &= \{\eta \in K | \eta \geq \tau \in L\} \\ Lk(L; K) &= \overline{St(L; K)} - St(\overline{L}; K) \end{aligned}$$

The order of a simplex $\tau$ in $K$, denoted by $Ord(\tau; K)$, measures the topological complexity of $\tau$ in $K$. Let $k = dim(St(\tau; K))$. $Ord(\tau; K)$ is the smallest integer $i$ such that there is a $(k-i)$ simplex $\eta$, in a suitable simplicial complex $F$, such that $St(\tau; K)$ and $St(\eta; F)$ are combinatorially equivalent.

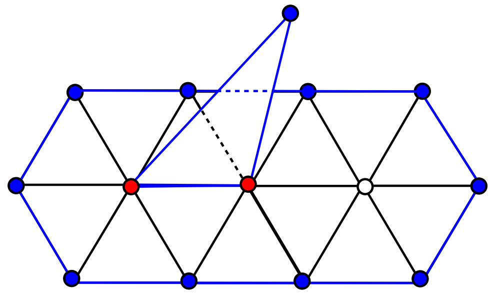


Figure 3.1: Order of a simplex is an indicator of the topological complexity of its star. Blue simplices are of order 1 and red vertices have order 2. Other simplices have order 0.

To illustrate order of a simplex, consider the 2D simplicial complex in Figure 3.1. The order of all triangles in the mesh is 0 because the star of a triangle in the mesh is the triangle itself. Hence $k = 2$. Now, we can choose a triangle with its faces as a simplicial complex so that the star of the triangle is combinatorially equivalent to a triangle in the mesh. Hence $k - i = 2$ and therefore $i = 0$. So all triangles of the mesh have order 0. The black edges are shared by two triangles. So $k = 2$. The star of a black edge is isomorphic to a triangle subdivided into two as shown in Figure 3.2. Thus order of black edges is also 0. Following a similar reasoning, all blue edges and vertices have order 1 and red vertices have order 2.

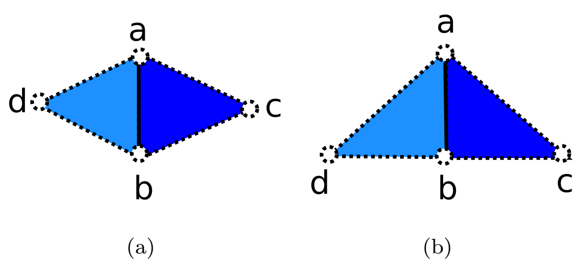



Figure 3.2: (a) Star of a black edge, say $ab$, contains exactly two triangles $abc$ and $abd$. (b) Triangle $acd$ subdivided into two is isomorphic with star of $ab$.

The *j-th boundary* of a simplicial complex $K$, denoted by $Bd_j K$ is the set of simplices with order greater than or equal to $j$. In Figure 3.1, the $0^{th}$ boundary is the entire mesh, the $1^{st}$ boundary is the set of blue edges, blue vertices and red vertices. The $2^{nd}$ boundary consists of the two red vertices.

For a k-simplex $\eta$ and a vertex $x$ that is affinely independent of the vertices $v_1, v_2, \ldots, v_{k+1}$ of $\eta$, the *cone* from $x$ to $\eta$ is defined as a simplex with vertices $x, v_1, v_2, \ldots, v_{k+1}$ and is denoted by $x \cdot \eta$.

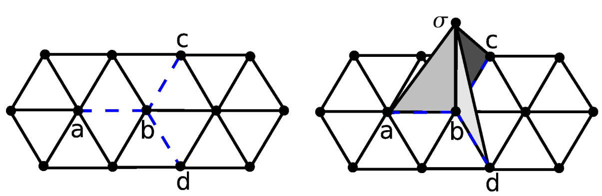
For each $i$, define $Bd_i^\omega K$ to be the simplicial complex formed by adding a dummy vertex $\omega$ and adding cones from $\omega$ to all simplices in $Bd_{i+1}K$. So, $Bd_i^\omega K = Bd_i K \cup (\omega \cdot Bd_{i+1}K)$. For a simplex $\eta \in Bd_i^\omega K$, we denote the link within $Bd_i^\omega K$ as $Lk_i^\omega(\eta; K)$.

We use edge contraction as the basic operation for mesh simplification. While doing edge contractions in a simplicial complex $K$, topology of $K$ would be preserved if a set of conditions, called *link conditions*, are satisfied. For an edge $ab$, the link conditions are:

$$Lk_i^{\omega}(a;K) \cap Lk_i^{\omega}(b;K) = Lk_i^{\omega}(ab;K) \quad \forall i \geq 0.$$

For a simplicial complex $K$, an embedded structure is a user defined subcomplex of $K$ where $dim(E) < dim(K)$. An extended complex, $\widetilde{K}$, as defined by [1, 2] is obtained from $K$ by introducing a dummy vertex $\sigma$ and adding cones from $\sigma$ to simplices in $E$ so that $\widetilde{K} = K \cup \sigma \cdot E$.

To illustrate this, consider the simplicial complex $K$ in Figure 3.3(a) where the edges in blue form the embedded structure. Then the extended complex is constructed by inserting cones from $\sigma$ to the blue edges as shown in Figure 3.3(b).




(a) (b)

Figure 3.3: (a) Simplicial complex $K$ with embedded structure in blue and (b) $\widetilde{K}$ obtained by inserting cones from $\sigma$.

# Chapter 4

# Topology Preservation

Let $K$ be a tetrahedral mesh with an embedded structure $E$ of dimension 2 or lower. $\widetilde{K}$ is the extended complex formed by adding cones to $E$ from a dummy vertex $\sigma$. Vivodtzev et al. [1, 2] assume that $K$ can be simplified without violating topology of $K$ or $E$ by contracting edges that satisfy link conditions of $\widetilde{K}$. They sketch a proof for the case $dim(K) = 2$. However, as indicated earlier, this proof has a major gap since it does not consider the cones added from $\sigma$ while analysing order of simplices in $\widetilde{K}$. Moreover, the proof does not extend to the case of $dim(K) = 3$. In this section, we present a proof for the case when $dim(K) = 3$. Analogous arguments prove the result for the case $dim(K) = 2$. Initially we assume that $E$ is disjoint from $Bd_1K$ and prove that link conditions are sufficient for topology preservation of $E$ and $K$. Later, we show that this assumption can be relaxed.

Consider edge contractions of $\widetilde{K}$ where the edges are not incident on the dummy vertex $\sigma$. The link conditions for a 3-complex $\widetilde{K}$ are:

$$Lk_0^\omega(a;\widetilde{K}) \cap Lk_0^\omega(b;\widetilde{K}) = Lk_0^\omega(ab;\widetilde{K}), \tag{III.0}$$

$$Lk_1^\omega(a;\widetilde{K}) \cap Lk_1^\omega(b;\widetilde{K}) = Lk_1^\omega(ab;\widetilde{K}), \tag{III.1}$$

$$Lk_2^\omega(a;\widetilde{K}) \cap Lk_2^\omega(b;\widetilde{K}) = \phi. \tag{III.2}$$

We want to prove that edge contractions that satisfy the above link conditions of $\widetilde{K}$ preserve topology of $K$ and $E$.

We adopt a two-step approach to prove this result. First, we show that topology of $K$ is preserved by proving that if an edge is selected for contraction then it will satisfy link conditions of $K$. This is done by proving the contrapositive statement – if an edge violates link conditions of $K$ then it will also violate link conditions of $\widetilde{K}$ and hence will not be selected for edge contraction.

Next, we show that topology of $E$ is preserved. For this, we classify edges into different categories based on whether they are part of $E$ or not. It is easy to show that edges outside $E$ that satisfy link conditions of $\widetilde{K}$ will not cause topology violation of $E$. For edges that belong to $E$, we use an approach similar to the one used for $K$ and show that link conditions of $\widetilde{K}$ are violated whenever link conditions of $E$ are violated.

## 4.1 Order of a simplex in $E$ and $\widetilde{K}$

To argue about violation of link conditions in $E$ and $\widetilde{K}$, it is important to understand the relationship between the order of a simplex in $E$ and its order in $\widetilde{K}$. The lemmas below state this relationship.

**Lemma 1.** *For an edge $ab \in E$, if $Ord(ab; E) = 1$, then*

*i.* $Ord(\sigma ab; \widetilde{K}) = 1$

*ii.* $Ord(ab; \widetilde{K}) \geq 1$.

**Lemma 2.** *For an edge $ab \in E$, if $Ord(ab; E) = 0$, then $Ord(\sigma ab; \widetilde{K}) = 0$.*

**Lemma 3.** *For a vertex $a \in E$, if $Ord(a; E) \geq 1$, then $Ord(\sigma a; \widetilde{K}) \geq 1$.*

**Lemma 4.** *For an edge $ab \in E$, if $Ord(ab; E) = 1$ and $Ord(a; E) = 2$, then $Ord(\sigma a; \widetilde{K}) = 2$.*

**Lemma 5.** $Bd_i^{\omega} K \subseteq Bd_i^{\omega} \widetilde{K}$, $\forall i \geq 0$.

The next section describes the proof for these lemmas. Readers not interested in the proof may skip to Section 4.3.

## 4.2 Proof of Lemmas

**Lemma 1.** *For an edge $ab \in E$, if $Ord(ab; E) = 1$, then*

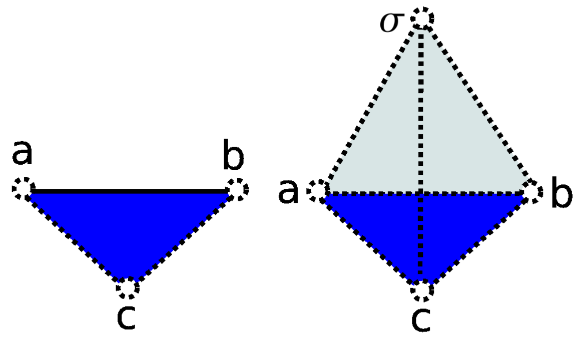
*i.* $Ord(\sigma ab; \widetilde{K}) = 1$

*ii.* $Ord(ab; \widetilde{K}) \geq 1$.

**Proof:** $Ord(ab; E) = 1$. So, $St(ab; E)$ has either exactly one triangle (as shown in Figure 4.1(a)) or more than two triangles and hence $St(\sigma ab; \widetilde{K})$ has either exactly one tetrahedron (Figure 4.1(b)) or more than two tetrahedra. Thus $Ord(\sigma ab; \widetilde{K}) = 1$. Since order of a face cannot be less than its coface, $Ord(ab; \widetilde{K}) \geq 1$.

**Lemma 2.** *For an edge $ab \in E$, if $Ord(ab; E) = 0$, then $Ord(\sigma ab; \widetilde{K}) = 0$.*

**Proof:** Since $Ord(ab; E) = 0$, $ab$ is either principal or incident on exactly two triangles (Figure 4.2(a)). Hence in $\widetilde{K}$, $\sigma ab$ is principal or incident on exactly two tetrahedra (Figure 4.2(b)) and therefore $Ord(\sigma ab; \widetilde{K}) = 0$.




(a) (b)

Figure 4.1: An order-1 edge $ab$ in $E$ and the corresponding order-1 triangle $\sigma ab$ in $\widetilde{K}$.

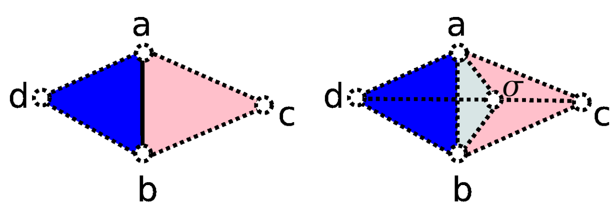
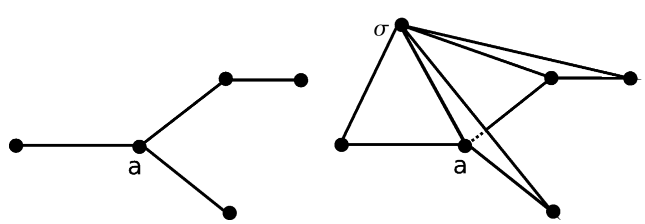



Figure 4.2: An order-0 edge $ab$ in $E$ and the corresponding order-0 triangle $\sigma ab$ in $\widetilde{K}$.

**Lemma 3.** *For a vertex $a \in E$, if $Ord(a; E) \geq 1$, then $Ord(\sigma a; \widetilde{K}) \geq 1$.*

**Proof:** Since $Ord(a; E) \neq 0$, $|St(a; E)|$ is not a manifold. Hence $|St(\sigma a; \widetilde{K})|$ is also not a manifold. Therefore $Ord(\sigma a; \widetilde{K}) \geq 1$.

**Lemma 4.** *For an edge $ab \in E$, if $Ord(ab; E) = 1$ and $Ord(a; E) = 2$, then $Ord(\sigma a; \widetilde{K}) = 2$.*




(a) (b)

Figure 4.3: (a) A 1-complex with a vertex $a$ on its 1-boundary. Note that all vertices with degree not equal to two, lie on the 1-boundary. (b) Add cones from $\sigma$ to the 1-complex. The edge $\sigma a$ is incident on three triangles and therefore lies on the 1-boundary of the 2-complex.

**Proof:**

`Case 1.` $a \in Bd_1 Bd_1 E$:

Since $Bd_1 E$ is a 1-complex, degree of $a$ in $Bd_1 E$ is different from 2 (see Figure 4.3(a)).

Since all the edges in $Bd_1E$ are order-1 edges in $E$, in particular, the edges incident on $a$ in $Bd_1E$ are order-1 edges in $E$. Hence by Lemma 1, the triangles in $\widetilde{K}$ formed by adding cones from $\sigma$ to these edges have order 1. Since in $Bd_1E$, $a$ is incident on one or more than two edges, the edge $\sigma a$ is incident on one or more than two triangles in $\widetilde{K}$(see Figure 4.3(b)). Hence $\sigma a \in Bd_1Bd_1\widetilde{K}$. We know from [9] that $Bd_1Bd_1\widetilde{K} \subseteq Bd_2\widetilde{K}$. Therefore $\sigma a \in Bd_2\widetilde{K}$. Thus $Ord(\sigma a; \widetilde{K}) = 2$.

`Case 2.` $a \notin Bd_1Bd_1E$:

Degree of $a$ in $Bd_1E$ is exactly 2. Besides the edge $ab$, assume that the vertex $a$ is incident on edge $ax$. Since $Ord(ab; E) = 1$, $ab$ is incident on either exactly one or more than two triangles in $E$. Let $ab$ be incident on exactly one triangle. Since all edges incident on $a$, besides $ax$ and $ab$, have order 0, the star of $a$ contains a half disk as shown in Figure 4.4. We now consider two cases based on the number of triangles incident on $ax$.

`Case 2a.` $ax$ is incident on one triangle:

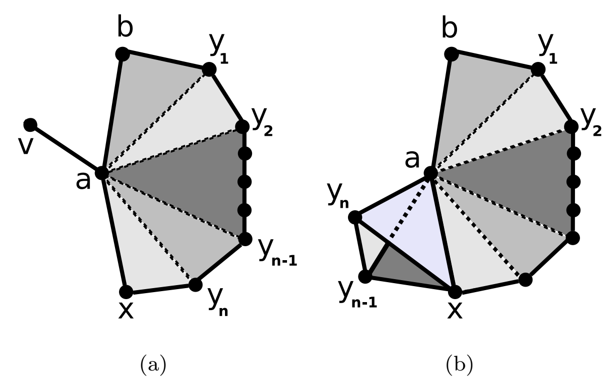



Figure 4.4: (a) $Ord(\sigma a; \widetilde{K}) = 2$ since $St(a; E)$ has a half disk and a principal edge. (b) $Ord(\sigma a; \widetilde{K}) = 2$ since $ab$ and $ax$ are incident on one and three triangles respectively.

In this case the sequence of triangles $aby_1, ay_1y_2, \ldots, ay_nx$ form a half disk of triangles. If $St(a; E)$ is exactly a half disk, then $Ord(a; E) = 1$. However, we know $Ord(a; E) = 2$, hence $St(a; E)$ must contain at least one principal edge in addition to the half disk, as shown in Figure 4.4(a). Corresponding to each triangle, $ay_iy_{i+1}$ in the half disk, $St(\sigma a; \widetilde{K})$ contains the tetrahedron $\sigma ay_iy_{i+1}$ and corresponding to each principal edge $av$, $St(\sigma a; \widetilde{K})$ contains the principle triangle $\sigma av$. It is not possible to have isomorphic subdivisions of $St(\sigma a; \widetilde{K})$ and star of a triangle in any arbitrary complex because of the principal triangles in $St(\sigma a; \widetilde{K})$. So, $Ord(\sigma a; \widetilde{K}) = 2$.

`Case 2b.` $ax$ is incident on more than two triangles:

In this case, $ab$ is incident on one triangle and $ax$ is incident on more than two triangles, as shown in Figure 4.4(b). Hence, $St(\sigma a; \widetilde{K})$ will contain the triangle $\sigma ab$, which is incident on only one tetrahedron, and $\sigma ax$ which is incident on more than two tetrahedra. No subdivision of star of a triangle in any arbitrary complex can contain triangles incident on one and more than two tetrahedra at the same time. Thus $Ord(\sigma a; \widetilde{K}) = 2$.

The above arguments extend to the case when $ab$ is incident on more than two triangles and $ab$ and $ax$ are incident on different number of triangles.

The only case remaining is when $ab$ and $ax$ do not form half disks but are incident on the same number of triangles. At least two of the triangles incident on $ab$ remain connected even when edge $ab$ is removed. This implies, at least two of the tetrahedra incident on $\sigma ab$ remain connected even when the triangle $\sigma ab$ is removed. Note that the above mentioned tetrahedra lie in $St(\sigma a; \widetilde{K})$. Now, consider a triangle in an arbitrary complex whose star contains at least three tetrahedra. One of the triangles in the subdivision of this star satisfies the property that its removal disconnects the incident tetrahedra. The existence of such a triangle implies that $St(\sigma a; \widetilde{K})$ and the star of a triangle cannot have isomorphic subdivisions. So, $Ord(\sigma a; \widetilde{K}) = 2$.

**Lemma 5.** *$Bd_i^\omega K \subseteq Bd_i^\omega \widetilde{K}$, $\forall i \geq 0$.*

**Proof:** Except for the simplices in $E$, all other simplices have the same star in $K$ and $\widetilde{K}$. Hence the only simplices whose order may differ between $K$ and $\widetilde{K}$ are those in $E$. For a simplex $s \in E$, $Ord(s; K) = 0$ since $E \cap Bd_1 K = \phi$. Since order of a simplex is a non-negative number, $Ord(s; \widetilde{K}) \geq 0$. Hence $Bd_i^\omega K \subseteq Bd_i^\omega \widetilde{K}$.

Note that the assumption, $E \cap Bd_1 K = \phi$, is essentially required only for the proof of Lemma 5.

## 4.3 Preserving topology of $K$

We now show that whenever link conditions are violated for an edge $ab$ of $K$, they are violated for $\widetilde{K}$ as well.

Suppose $i^{th}$ link condition (where $i = 0, 1, 2$) is violated in $K$. Then there is a simplex $s$ in $Bd_i^\omega K$ such that $s \in Lk_i^\omega(a; K)$, $s \in Lk_i^\omega(b; K)$ and $s \notin Lk_i^\omega(ab; K)$. By Lemma 5, $s \in Lk_i^\omega(a; \widetilde{K})$ and $s \in Lk_i^\omega(b; \widetilde{K})$. We show that $s \notin Lk_i^\omega(ab; \widetilde{K})$.

For a simplex $s \in E$, since $E \cap Bd_1 K = \phi$, $Ord(s; K) = 0$. When $K$ is extended to $\widetilde{K}$, cones are added from $\sigma$ to simplices in $E$. The cones thus added to $s$ may increase the order of $s$ in $\widetilde{K}$ i.e., $Ord(s; \widetilde{K})$ may be greater than $Ord(s; K)$. Thus a new simplex that appears in $Lk_i^\omega(ab; \widetilde{K})$ is a simplex belonging to one of the following types:

I. a cone from $\sigma$

II. a cone from $\omega$ to simplices incident on $\sigma$

III. a cone from $\omega$ to simplices in $E$

IV. the subcomplex $E$ (if $i > 0$).

If any of these new simplices in $Lk_i^\omega(ab; \widetilde{K})$ is $s$, link conditions in $\widetilde{K}$ would be satisfied. However, we show that none of the new simplices lie in $Bd_i^\omega K$ and hence cannot be $s$ since $s \in Bd_i^\omega K$. Type I or type II is not in $Bd_i^\omega K$ since the dummy vertex $\sigma \notin Bd_i^\omega K$. Type III is not in $Bd_i^\omega K$ because in $Bd_i^\omega K$, cones from $\omega$ are added only to simplices whose order is higher than 0. However, all simplices in $E$ have order 0 in $K$. Type IV is not in $Bd_i^\omega K$ because all simplices in $E$ have order 0 and hence do not belong to $Bd_i^\omega K$, $i > 0$. Hence, none of the new simplices in $Lk_i^\omega(ab; \widetilde{K})$ can be $s$.

Thus the simplex $s$ lies in both $Lk_i^\omega(a; \widetilde{K})$ and $Lk_i^\omega(b; \widetilde{K})$ but does not lie in $Lk_i^\omega(ab; \widetilde{K})$. So $i^{th}$ link condition of $\widetilde{K}$ is violated.

## 4.4 Preserving topology of $E$

An edge $ab$ either belongs to the subcomplex $E$ or lies outside $E$. In either case we show that if contracting $ab$ does not violate link conditions in $\widetilde{K}$ then the topology of $E$ is preserved.

### 4.4.1 Contracting edges not in $E$

Let $ab$ denote an edge in $\widetilde{K}$ that is not contained in $E$. Let $c$ be the new vertex obtained after contracting $ab$. Let $F$ denote the embedded structure after contracting $ab$. A simplex $<v_1, \ldots, v_n, c>$ lies in $F$ if and only if either $<v_1, \ldots, v_n, a>$ or $<v_1, \ldots, v_n, b>$ lies in $E$.

`Case 1.` $a, b \notin E$:

None of the vertices of $E$ are affected by the contraction. So $F = E$ and topology of $E$ is preserved.

`Case 2.` $a \in E$; $b \notin E$:

After contracting $ab$ all simplices $<v_1, \ldots, v_n, a> \in E$ become $<v_1, \ldots, v_n, c> \in F$. This renaming of a vertex does not change the topology of $E$.

`Case 3.` $a, b \in E$; $ab \notin E$:

This edge contraction is rejected because link condition (III.0) is violated. Cones are added from $\sigma$ to $a$ and $b$ but not to $ab$. So, $\sigma \in Lk_0^\omega(a; \widetilde{K}) \cap Lk_0^\omega(b; \widetilde{K})$ but $\sigma \notin Lk_0^\omega(ab; \widetilde{K})$.

### 4.4.2 Contracting edges in $E$

The embedded structure $E$ is a 2-complex. The two link conditions corresponding to $Bd_0^\omega E$ and $Bd_1^\omega E$ are:

$$Lk_0^\omega(a;E) \cap Lk_0^\omega(b;E) = Lk_0^\omega(ab;E), \tag{II.0}$$

$$Lk_1^\omega(a;E) \cap Lk_1^\omega(b;E) = \phi. \tag{II.1}$$

We show that if the $i^{th}$ link condition, $(i = 0, 1)$, is violated in $E$ then $i^{th}$ link condition in $\widetilde{K}$ is also violated. First, we show that if $ab \in Bd_i E$, then $ab \in Bd_i \widetilde{K}$, so that it is meaningful to talk about contracting $ab$ in $i^{th}$ boundary of $\widetilde{K}$.

It is clear that $ab \in Bd_0 \widetilde{K}$ since $ab \in E \subset K \subset \widetilde{K}$. Now, suppose $ab \in Bd_1 E$. The order of an edge in a 2-complex cannot be two or higher. So, $Ord(ab;E) = 1$. Using Lemma 1, $ab \in Bd_1 \widetilde{K}$. Thus, if $ab \in Bd_i E$, then $ab \in Bd_i \widetilde{K}$.

Now, suppose $i^{th}$ link condition is violated in $E$ while contracting edge $ab$. Then there is a simplex $s \in Bd_i^\omega E$ such that $s \in Lk_i^\omega(a;E)$, $s \in Lk_i^\omega(b;E)$ and $s \notin Lk_i^\omega(ab;E)$.

`Case 1.` Link condition (II.0) is violated:

Since $Bd_0^\omega E$ is a 2-complex, $s$ is either an edge or a vertex.

`Case 1a.` $s$ is an edge, say $v_1v_2$:

If $v_1 \neq \omega$ and $v_2 \neq \omega$, then $av_1v_2 \in E$ and hence $\sigma a v_1 v_2 \in \widetilde{K}$. So, $\sigma v_1 v_2 \in Lk_0^\omega(a;\widetilde{K})$. Similarly, $\sigma v_1 v_2 \in Lk_0^\omega(b;\widetilde{K})$. Therefore, $Lk_0^\omega(a;\widetilde{K}) \cap Lk_0^\omega(b;\widetilde{K})$ contains a triangle.

Now, assume that either $v_1$ or $v_2$ is the dummy vertex $\omega$. Without loss of generality, let $v_1$ be the dummy vertex. Since cones from $\omega$ are added only to simplices in $Bd_1 E$, $Ord(av_2;E) = 1$. Using Lemma 1, $Ord(\sigma a v_2;\widetilde{K}) = 1$ and so $\omega\sigma a v_2 \in Bd_0^\omega \widetilde{K}$. Hence $\omega\sigma v_2 \in Lk_0^\omega(a;\widetilde{K})$. Similarly, $\omega\sigma v_2 \in Lk_0^\omega(b;\widetilde{K})$. Therefore, $Lk_0^\omega(a;\widetilde{K}) \cap Lk_0^\omega(b;\widetilde{K})$ contains a triangle.

In both cases the link condition (III.0) is violated because $Lk_0^\omega(ab;\widetilde{K})$ cannot contain a triangle.

`Case 1b.` $s$ is a vertex, say $v$:

If $v \neq \omega$, then $av, bv \in E$ and $abv \notin E$ which implies $\sigma av, \sigma bv \in Bd_0^\omega \widetilde{K}$ and $\sigma abv \notin Bd_0^\omega \widetilde{K}$. Hence $\sigma v \in Lk_0^\omega(a;\widetilde{K}) \cap Lk_0^\omega(b;\widetilde{K})$ and not in $Lk_0^\omega(ab;\widetilde{K})$. Thus link condition (III.0) is violated.

If $v$ is $\omega$, then since link condition (II.0) is violated, $\omega \notin Lk_0^\omega(ab;E)$. This implies $\omega ab \notin Bd_0^\omega E$, and therefore $Ord(ab;E) = 0$. Since $\omega \in Lk_0^\omega(a;E)$, it follows that $Ord(a;E) \geq 1$, because cones from $\omega$ are added only to simplices in $Bd_1 E$. Using Lemma 3, $Ord(\sigma a;\widetilde{K}) \geq 1$.

Hence $\omega\sigma a \in Bd_0^\omega \widetilde{K}$ and $\omega\sigma \in Lk_0^\omega(a;\widetilde{K})$. Similarly, $\omega\sigma \in Lk_0^\omega(b;\widetilde{K})$. However, since $Ord(ab;E) = 0$, it follows from Lemma 2, that $Ord(\sigma ab;\widetilde{K}) = 0$ and hence $\omega\sigma \notin Lk_0^\omega(ab;\widetilde{K})$. Therefore link condition (III.0) is violated.

`Case 2.` Link condition (II.1) is violated:

Since $Bd_1^\omega E$ is a 1-complex, $s$ is a vertex.

`Case 2a.` $s$ is a vertex, $v \in E$:

Since $Bd_1^\omega E$ is a 1-complex, $Ord(av;E) = 1$. Using Lemma 1, $Ord(\sigma av;\widetilde{K}) = 1$, which implies $\sigma v \in Lk_1^\omega(a;\widetilde{K})$. Similarly, $\sigma v \in Lk_1^\omega(b;\widetilde{K})$. Therefore, $Lk_1^\omega(a;\widetilde{K}) \cap Lk_1^\omega(b;\widetilde{K})$ contains an edge.

`Case 2b.` $s$ is $\omega$:

Since in $Bd_1^\omega E$, cones from $\omega$ are added only to vertices of order 2, $Ord(a;E) = 2$. Using Lemma 4, $Ord(\sigma a;\widetilde{K}) = 2$. Hence, $\omega\sigma a \in Bd_1^\omega \widetilde{K}$ and $\omega\sigma \in Lk_1^\omega(a;\widetilde{K})$. Similarly, $\omega\sigma \in Lk_1^\omega(b;\widetilde{K})$. Therefore, $Lk_1^\omega(a;\widetilde{K}) \cap Lk_1^\omega(b;\widetilde{K})$ contains an edge.

In both cases the link condition (III.1) is violated because $Lk_1^\omega(ab;\widetilde{K})$ cannot contain an edge.

## 4.5 2-complexes

We now consider the analogous problem in 2D, where $K$ is a 2-complex and $E$ is a 1-complex disjoint from $Bd_1K$. Vivodtzev et al. [2] describe a proof of topology preservation by claiming that $Bd_1\widetilde{K} = E \cup Bd_1K$. However, this is not true because $Bd_1\widetilde{K}$ contains edges incident on $\sigma$ in addition to edges in $E \cup Bd_1K$. For example, in Figure 3.3(b), the edges $\sigma a$, $\sigma b$, $\sigma c$ and $\sigma d$ are all order 1 edges.

Moreover, the arguments used in their proof infers order of an edge by considering the number of triangles shared by the edge. These arguments do not extend to the case of 3-complexes. For instance, the star of an edge in 3-complexes contains several tetrahedra but the number of tetrahedra do not directly indicate the order of the edge. Our arguments look at relationship between the star of simplices in $E$ and $\widetilde{K}$ to infer the order of a simplex. This approach is more general than arguments in [2]. Hence we prove the result in the case of 2D meshes using arguments analogous to the 3D case.

Topology preservation of $K$ and topology preservation of $E$ while contracting edges that do not lie in $E$, can be shown using the same arguments as in the 3D case. For the sake of completeness, we describe topology preservation of $E$ when an edge from $E$ is contracted, although this is analogous to the 3D case.

Since $E$ is a 1-complex, $E$ has one link condition given by:

$$Lk_0^\omega(a;E) \cap Lk_0^\omega(b;E) = \phi. \tag{I.0}$$

We show that if contracting an edge $ab \in E$ violates link condition (I.0), then link conditions are violated in $\widetilde{K}$ as well. Since $E$ is a 1-complex, only vertices can be present in $Lk_0^\omega(a;E) \cap Lk_0^\omega(b;E)$. The vertex, $v$, that violates link condition (I.0) can either be the dummy vertex $\omega$ or a vertex in $E$.

`Case 1.` $v \in E$:

Since $av \in E$, $\sigma av \in \widetilde{K}$. Thus, $\sigma v \in Lk_0^\omega(a;\widetilde{K})$. Similarly, $\sigma v \in Lk_0^\omega(b;\widetilde{K})$. Therefore, $Lk_0^\omega(a;\widetilde{K}) \cap Lk_0^\omega(b;\widetilde{K})$ contains an edge.

`Case 2.` $v$ is $\omega$:

Since cones from $\omega$ are added only to vertices of order 1, $Ord(a;E) = 1$. This means degree of $a$ in $E$ is either exactly one or more than two. Hence in $\widetilde{K}$, the edge $\sigma a$ is incident on either exactly one or more than two triangles. Thus $Ord(\sigma a;\widetilde{K}) = 1$. Hence $\omega\sigma a \in Bd_0^\omega \widetilde{K}$ and $\omega\sigma \in Lk_0^\omega(a;\widetilde{K})$. Similarly, $\omega\sigma \in Lk_0^\omega(b;\widetilde{K})$. Therefore, $Lk_0^\omega(a;\widetilde{K}) \cap Lk_0^\omega(b;\widetilde{K})$ contains an edge.

Since $\widetilde{K}$ is a 2-complex, $Lk_0^\omega(ab;\widetilde{K})$ cannot contain an edge. Hence, in both the cases $0^{th}$ link condition is violated in $\widetilde{K}$.

## 4.6 Embedded structure on boundary

We now extend our result to include embedded structures that intersect the boundary of $K$. This assumption is a serious shortcoming of the previous result [1, 2] because embedded structure often lies on the boundary, as shown in our applications.

Lemmas 1-4 clearly show that the relation between order of a simplex in $E$ and $\widetilde{K}$ does not change whether embedded structure intersect the boundary of $K$ or not. However, Lemma 5 is not necessarily true when we remove the assumption on $E$. For example, if $E$ contains a triangle, say $abc$ which is part of a single tetrahedron in $K$, then $Ord(abc;K) = 1$. However, the triangle $abc \in \widetilde{K}$ is incident on two tetrahedra due to the cone from $\sigma$ and hence $Ord(abc;\widetilde{K}) = 0$. Thus $Bd_1^\omega K \not\subseteq Bd_1^\omega \widetilde{K}$.

However, note that Lemma 5 is not necessary to show that topology of $E$ is preserved while contracting an edge. A simple solution that ensures that topology of $K$ is also preserved is to verify that link conditions of $K$ are satisfied in addition to the link conditions for $\widetilde{K}$. Although this additional validation seems to require evaluation of three new link conditions, in practice, the link conditions for $K$ and $\widetilde{K}$ can be verified in a single procedure.

## 4.7 Necessity of Link Conditions

For a manifold $M$, the link condition for an edge $ab$ is $Lk(a;M) \cap Lk(b;M) = Lk(ab;M)$. The results from [9] show that for 2-manifolds and 3-manifolds, link conditions are necessary for ensuring topology preservation. The assumption of the domain being a manifold is very restrictive and does not hold in many practical situations. However, in a more general setting of non-manifolds, the link conditions are not necessary for ensuring topology preservation. In this section, we show that under practically reasonable assumptions, the link conditions are almost always necessary. In particular, when $K$ is a 2-manifold with or without boundary, $E$ is a 1-manifold with or without boundary, and $E \cap Bd_1 K = \phi$, we show that except for two special cases, the link conditions of $\widetilde{K}$ are necessary for preserving the topology of $K$ and $E$.

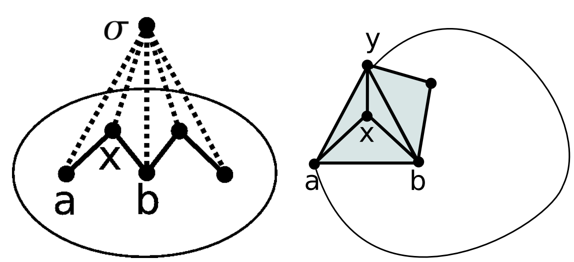
Since necessity of link conditions for topology preserving simplification of meshes with embedded structures were not studied earlier, it was not clear if such simplifications were too conservative and disallowed edge contractions that preserved the topology but violated the link conditions. Our result below shows that 2D mesh simplification with embedded structures is nearly optimal using link conditions, in the sense that most of the edge contractions that we disallow do in fact modify topology.

The link conditions for $\widetilde{K}$,

$$Lk_0^{\omega}(a; \widetilde{K}) \cap Lk_0^{\omega}(b; \widetilde{K}) = Lk_0^{\omega}(ab; \widetilde{K}), \tag{4.1}$$

$$Lk_1^{\omega}(a; \widetilde{K}) \cap Lk_1^{\omega}(b; \widetilde{K}) = \phi. \tag{4.2}$$

are not necessary in the following special cases.




(a) (b)

Figure 4.5: Two exceptions where the edge $ab$ can be contracted without violating topology, but violates the link conditions. (a) Exception 1 : $a$, $x$ and $b$ are the first 3 vertices in $E$. (b) Exception 2 : $ya$ is on the boundary of $K$.

`Exception 1.` $Lk_0^{\omega}(a; \widetilde{K}) \cap Lk_0^{\omega}(b; \widetilde{K})$ contains an edge $\sigma x$ where $a$, $x$ and $b$ are three vertices of $E$ as shown in Figure 4.5(a). In this case, there is no topology violation since the complex before and after contraction is topologically equivalent. However, link condition (4.1) is violated as an edge is present in $Lk_0^{\omega}(a; \widetilde{K}) \cap Lk_0^{\omega}(b; \widetilde{K})$.

`Exception 2.` $Lk_0^{\omega}(a; \widetilde{K}) \cap Lk_0^{\omega}(b; \widetilde{K})$ contains an edge $xy$ such that either $ya$ or $yb$ lies on the boundary of $K$ (Figure 4.5(b)). If $ya$ and $yb$ were not on the boundary of $K$, then after contraction of $ab$, the edge $yc$ would have been incident on three triangles and hence caused a topology violation.

However, without loss of generality, say $ya$ is on the boundary. After contraction, $yc$ is incident on two triangles and hence there is no topology violation.

We now show that, in all other cases, whenever link conditions of $\widetilde{K}$ are violated, the topology of $K$ or $E$ changes. The link conditions of $\widetilde{K}$ could be violated in three ways:

`Case 1.` An edge $xy$ violates link condition (4.1):

`Case 1a.` $xy \in K$:

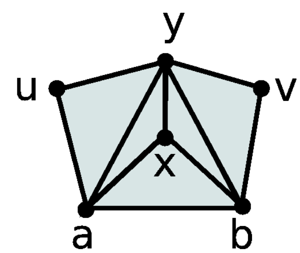



Figure 4.6: After contracting $ab$ to $c$, $yc$ becomes part of three triangles

Since Exception 2 is excluded, the edge $ya$ is part of two triangles in $K$ - $yax$ and $yau$. Similarly, $yb$ is part of two triangles in $K$ - $ybx$ and $ybv$ (Figure 4.6). If $u \neq v$, after contracting $ab$ to $c$, the edge $yc$ is part of three triangles, $ycx$, $ycu$ and $ycv$. In $K$, no edge is part of three triangles. Thus contracting $ab$ results in change in topology of $K$. If $u = v$, then the same argument holds after replacing $xy$ with $yu$.

`Case 1b.` $xy \notin K$:

The only edges in $Bd_0^{\omega}\widetilde{K}$ that are not part of $K$ are the edges incident on either $\sigma$ or $\omega$. Since all the edges in $E$ are part of three triangles in $\widetilde{K}$, the order of any edge in $E$ is 1 in $\widetilde{K}$. Hence for any simplex $s \in Bd_0^{\omega}\widetilde{K}$, if cone from $\sigma$ is added to $s$, then cone from $\omega$ is also added to $s$. So we can assume without loss of generality that $x$ is $\omega$.

If $xy$ is $\omega\sigma$, since cones from $\omega$ are added only to simplices of order 1 or higher $Ord(\sigma a; E) = Ord(\sigma b; E) = 1$. This means that $\sigma a$ and $\sigma b$ are incident on a single triangle. Hence $a$ and $b$ are degree-1 vertices in $E$. If the path from $a$ to $b$ in $E$ consists of only one edge, then contracting $ab$ would destroy the embedded structure. If there are more than one edge, since Exception 1 is excluded, contracting $ab$ would create a new cycle in $E$ (Figure 4.9). Thus topology of $E$ is violated.

The only other edges whose order is 1 in $\widetilde{K}$ are the edges in $E$ or the edges in $Bd_1K$. Hence, $ya$ and $yb$ are either both in $E$ or $Bd_1K$. Note that it is not possible for one edge to be in $E$ and the other to be in $Bd_1K$ since $E \cap Bd_1K = \phi$.

If $ya$, $yb \in E$, since Exception 1 is excluded, either $ab \in E$ or there exists edges $wa$, $bz \in E$. On the other hand, if $ya$, $yb \in Bd_1K$, since $Bd_1K$ is a 1-manifold, either $ab \in Bd_1K$ or there exist edges $wa$, $bz \in Bd_1K$.

If $ab \in Bd_1K$, then contracting $ab$ changes topology of $Bd_1K$ because the cycle $ab - by - ya$ exists before contraction and does not exist after contraction. The same argument holds if $ab \in E$. So, we can assume that the path $wa - ay - yb - bz$ exists in either $Bd_1K$ or $E$ (Figure 4.7). After contracting $ab$ to $c$, the edges $wc$, $yc$ and $zc$ are incident on $c$ and thus either $Bd_1K$ or $E$ becomes a non-manifold after contraction. If $w = z$, then a cycle is destroyed.

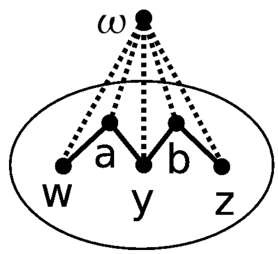



Figure 4.7: $Bd_1K$ or $E$ becomes a non-manifold after contracting $ab$

`Case 2.` Link condition (4.1) is violated only by vertices, say vertex $x$:

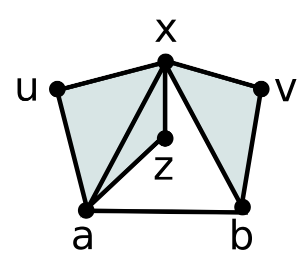



Figure 4.8: If $ab$ contracts to $c$, the edge $xc$ would be incident on three triangles.

If $x$ is not $\sigma$ or $\omega$, then edge $ax$, $bx$ and $ab$ exist, but the triangle $axb$ does not exist. Since $K$ cannot have principal edges, there exists triangles $axu$ and $bxv$. We can assume that $u \neq v$, since $Lk_0^\omega(a;\widetilde{K}) \cap Lk_0^\omega(b;\widetilde{K})$ does not contain any edges. If both $ax$ and $bx$ were part of exactly one triangle, then $Lk_0^\omega(a;\widetilde{K}) \cap Lk_0^\omega(b;\widetilde{K})$ would have contained the edge $\omega x$. Hence we can assume that $ax$ is shared by a second triangle, $axz$ (Figure 4.8). After contracting $ab$ to $c$, the edge $xc$ becomes part of three triangles, $xzc$, $xcu$ and $xcv$ and thus topology of $K$ changes.

If $x$ is either $\sigma$ or $\omega$, then $a$ and $b$ are vertices in $E$ or $Bd_1K$. We can assume without loss of generality that $x$ is $\omega$ since all the cones added from $\sigma$ are also added from $\omega$. If there is no path from $a$ to $b$ in link of $\omega$ then contracting $ab$ would violate topology as it would connect two components which are disconnected in $Bd_1\widetilde{K}$. If there is a path from $a$ to $b$ in link of $\omega$,

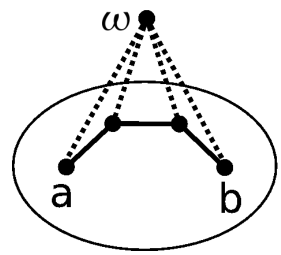



Figure 4.9: Contracting edge $ab$ creates a new cycle in $Bd_1K$ or $E$.

then since $Lk_0^{\omega}(a;\widetilde{K}) \cap Lk_0^{\omega}(b;\widetilde{K})$ contains no edges, the path contains at least 3 edges (Figure 4.9). Otherwise if $ay$, $yb$ are the only two edges, then edge $\omega y \in Lk_0^{\omega}(a;\widetilde{K}) \cap Lk_0^{\omega}(b;\widetilde{K})$. Hence contracting $ab$ introduces a new cycle and changes topology of $E$ or $Bd_1K$.

`Case 3.` Link condition (4.2) is violated by a vertex $x$:

If $x \in K$, then the cycle $ax$, $bx$ and $ab$ exists in $E \cup Bd_1K$ and the cycle would be destroyed by contracting $ab$ thus changing the topology of $E$ or $Bd_1K$. If $x$ is $\omega$ or $\sigma$, $a$ and $b$ are degree-1 vertices in $E$. We have already considered this case earlier in Case 1b.

# Chapter 5

# Implementation

Our implementation of simplification of 3D meshes with embedded structures essentially contracts edges that satisfy the link conditions. The input mesh represents a 3-manifold with or without boundary and the embedded structure is a 2-manifold with or without boundary. The simplification proceeds until the number of vertices in the mesh reaches a user-specified threshold, $v$, or until no edge can be contracted without violating topology, whichever happens earlier.

## 5.1 Data structure and algorithm

We represent the input mesh using the triangle-edge data structure [14]. Each triangle has a flag to identify if the triangle belongs to $E$. Using ideas from [8], we modify the quadric error metric [6] to handle 3D meshes with a scalar field and improve the quality of the mesh. In order to ensure that the geometry of the embedded structure is minimally affected, triangles in the embedded structure are treated similar to boundary triangles i.e. edges that are incident on the embedded structure are penalized with a higher weight. Edges are selected from a priority queue in the order of increasing cost. Function SIMPLIFY describes the high level algorithm. Implementation details, except for the evaluation of order of simplices in $\widetilde{K}$, can be found in earlier work [6, 8].

SIMPLIFY($K$)

Initialise priority queue $Q$ with edges in $K$

while(# of vertices $> v$ and Q.notempty()) do

$ab = Q.pop()$

if(Link Conditions for $K$ and $\widetilde{K}$ are satisfied)

Contract $ab$ and update $K$ and $Q$

endif

endwhile

## 5.2 Evaluating the order of simplices in $\widetilde{K}$

Simplices that are not part of $E$ have the same order in $K$ and $\widetilde{K}$. Hence we consider only simplices whose order is different in $K$ and $\widetilde{K}$ in this section.

**Tetrahedra:** All tetrahedra incident on $\sigma$ have order 0.

**Triangles:** A triangle $abc \in E \cap Bd_1K$ is incident on exactly two tetrahedra, one in $K$ and the other a cone from $\sigma$, and hence $Ord(abc; \widetilde{K})$ is 0. If $abc$ lies in the interior of $K$ then $abc$ is incident on three tetrahedra and hence has order 1. The triangle could also be a cone from $\sigma$, namely $\sigma ab$. The triangle $\sigma ab$ is incident on two tetrahedra if $St(ab; E)$ has two triangles. In this case, $Ord(\sigma ab; \widetilde{K})$ is 0. If $St(ab; E)$ has exactly one triangle then $Ord(\sigma ab; \widetilde{K})$ is 1.

**Edges:** For an edge $\sigma a$, $Ord(\sigma a; \widetilde{K}) = Ord(a; E)$. For edges in $E$, we consider the following two cases.

`Case 1.` Edge $ab \in E$ lies on $Bd_1K$:

If $ab$ is incident on exactly two triangles $abc$ and $abd$ in $E$ and if both $abc$ and $abd$ are on $Bd_1K$ as shown in Figure 5.1(a), then $Ord(ab; \widetilde{K})$ is 0. This is because the half sphere $St(ab; K)$ grows to become a sphere in $\widetilde{K}$ after adding the cones from $\sigma$. If, at least one triangle, say $abc$ is in the interior of $K$, as shown in Figure 5.1(b) then $Ord(ab; \widetilde{K})$ is 2. This is because $Bd_1K$ is a 2-manifold and hence $ab$ is incident on a triangle $abx$ on $Bd_1K$. Triangle $abx$ is incident on only one tetrahedron and $abc$ on three tetrahedra. Subdividing the star of a triangle cannot create two triangles, one of which is incident on exactly one tetrahedron and the other on three tetrahedra.

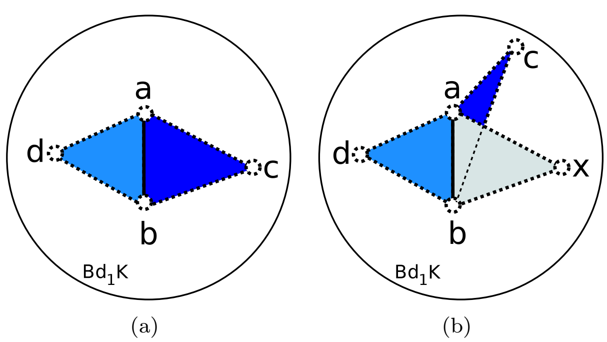
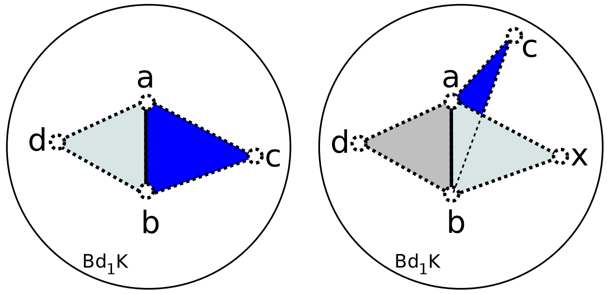



Figure 5.1: (a) $St(ab; E)$ contains triangles $abc$ and $abd$, both lying in $Bd_1K$. (b) $St(ab; E)$ contains triangle $abd$ on $Bd_1K$ and $abc$ in the interior of $K$.

If $ab$ is incident on only one triangle $abc \in E$ and $abc$ lies on $Bd_1K$, as shown in Figure 5.2(a), then $Ord(ab; \widetilde{K}) = 1$, since $St(ab; \widetilde{K})$ is a half sphere. If $abc$ lies in the interior of $K$, as shown



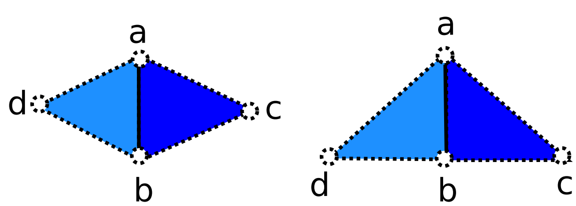

(a) (b)

Figure 5.2: (a) $St(ab;E)$ contains exactly one triangle $abc$ lying in $Bd_1K$. (b) $St(ab;E)$ contains exactly one triangle $abc$ lying in the interior of $K$.

in Figure 5.2(b), then $Ord(ab;\widetilde{K}) = 2$, due to triangles $abx$ and $abc$ incident on one and three tetrahedra respectively, as before.

`Case 2.` Edge $ab \in E$ lies outside $Bd_1K$:

$Ord(ab;\widetilde{K}) = 1$, if $St(ab;E)$ has exactly two triangles because $St(ab;\widetilde{K})$ has isomorphic subdivision with star of a triangle $acd \in E$ in the interior of $K$, as shown in Figure 5.3.



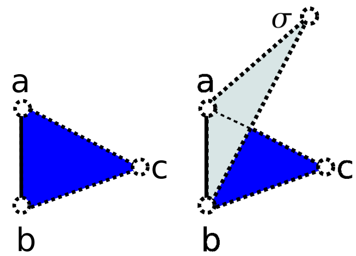

(a) (b)

Figure 5.3: (a) $St(ab;E)$ contains exactly two triangles $abc$ and $abd$ in the interior of $K$. (b) Isomorphic subdivision of star of $acd$.

If $St(ab;E)$ has only one triangle $abc$, as shown in Figure 5.4(a) then $Ord(ab;\widetilde{K})$ is 2 since $\sigma ab$ is incident on exactly one tetrahedron, while $abc$ is incident on three tetrahedra, as shown in Figure 5.4(b).

**Vertices:** For a vertex $a \notin Bd_1K$, $Ord(a;\widetilde{K}) = Ord(a;E){+}1$, using isomorphic subdivisions of $St(ab;E)$ described in Figure 5.3(a) and Figure 5.4(a).

For a vertex $a \in Bd_1K$, if $St(a;E) \subset Bd_1K$, then $Ord(a;\widetilde{K}) = Ord(a;E)$, using the subdivision of $St(ab;K)$, described in Figure 5.1(a) and Figure 5.2(a). If $St(a;E)$ is a disk such that a half disk lies on $Bd_1K$ and the remaining half disk lies in the interior of $K$, then $Ord(a;\widetilde{K})$ is 2, using the subdivision of $St(ab;K)$, described in Figure 5.1(b). If $St(a;E)$ is a half disk that lies in the interior of $K$, then




(a) (b)

Figure 5.4: (a) $St(ab;E)$ contains exactly one triangle $abc$ in the interior of $K$. (b) $\sigma ab$ is incident on one tetrahedron, whereas $abc$ is incident on three tetrahedra.

$Ord(a;\widetilde{K})$ is 2, using the subdivision of $St(ab;K)$, described in Figure 5.2(b). Although it is possible to analyse the remaining cases when $a$ lies on $Bd_1K$ and its star lies partially in the interior of $K$, for ease of computation and book keeping involved during implementation, we consider such a vertex to have order 3. Overestimating the order of the vertex assures that the topology is preserved at the cost of preventing a few legal edge contractions.

## 5.3 Updating priority queue after an edge contraction

After contracting an edge $ab$ to $c$, cost of the edges in the neighbourhood of $c$ have to be recomputed in order to update the priority queue. This is a computationally expensive step. Moreover, it is possible that some of these computations are redundant since the cost of an edge will be recomputed several times during a sequence of edge contractions.

We apply a simple optimisation to reduce the overhead involved in updating the priority queue after an edge contraction. Associated with each vertex $a$, we store a timestamp that indicates when any edge incident on $a$ was last contracted. Each edge $ab$ in the priority queue also has a timestamp that indicates when the edge $ab$ was added to the priority queue. If the timestamp associated with edge $ab$ is older than those associated with vertices $a$ or $b$, it means that the current cost of the edge $ab$ is stale and has to be recomputed. However, we do this recomputation in a lazy manner. Whenever an edge $ab$ is selected for contraction, we recompute the cost of the edge if it is stale. The edge is inserted back to the priority queue with a new timestamp and the process is repeated till a non-stale edge is selected for contraction.

The one disadvantage with this scheme is that by lazily updating the priority queue we may not always be selecting the edge with least cost for contraction. However, note that an edge which has a high cost before contraction is very likely to have a high cost even after the contraction. Hence, even though this optimisation may not always select the edge with least cost, in most cases it will be selecting a low cost edge for contraction and hence will not affect the quality of simplification a lot.

# Chapter 6

# Applications

## 6.1 Isosurface Topology Preservation

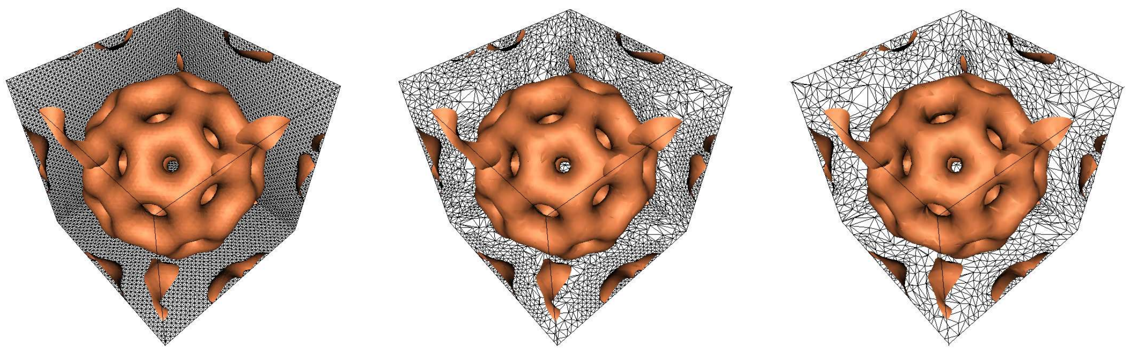
Topology of isosurfaces gives insights about important features of the underlying volumetric data. The topology of isosurfaces may be important in medical applications like cortex labelling, organ template fitting etc. In CAD modelling, features like tunnels and holes are used in identifying important characteristics of the model.

By treating the isosurface as an embedded structure, we ensure topology preservation of the isosurface while doing mesh simplification. For this purpose, we extract the relevant isosurface and then do remeshing of the original tetrahedral mesh so that the triangles which make up the isosurface become faces of the tetrahedra in the new mesh. To do this, we consider each tetrahedron that intersects the isosurface and subdivide the tetrahedron based on how the isosurface passes through the tetrahedron. Figure 6.1 shows that the topology of isosurface is preserved after simplification of the Bucky Ball dataset. Moreover, the geometry of the embedded structure remain close to the original. A simple extension allows us to simplify the mesh while preserving the topology of multiple isosurfaces.

## 6.2 Molecular Surface Topology Preservation

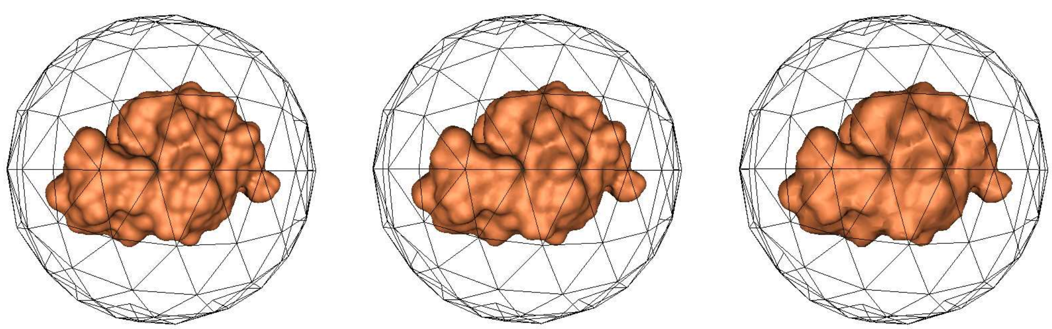
Modelling of molecular surfaces of proteins is useful in applications like biomolecular recognition, study of drug binding cavities etc. Preserving the topology of the molecular surface is important in studying the properties of the molecule. For example, the stability of a protein depends on the number and size of voids [18].

A molecule in solution is represented by a volume mesh. We simplify the mesh while preserving the topology of the molecular surface, which is specified as an embedded structure. Figure 6.2 shows a molecular surface before and after simplification.



(a) (b) (c)

Figure 6.1: Isosurface topology preserving volume simplification. From left to right - isosurface at 100%, 30% and 10% of original data.



(a) (b) (c)

Figure 6.2: Molecular surface simplification. The mesh representing the volume occupied by the protein molecule (PDB ID: 193L) is simplified while preserving the topology of the molecular surface. From left to right - molecular surface at 100%, 30% and 10% of original data.

# Chapter 7

# Results

We report the results of our experiments on four datasets. The first three datasets from the AIM@SHAPE repository have a scalar field associated with them. In each case, we are interested in preserving the topology of one or more isosurfaces extracted from the data. Error introduced in the scalar field due to the simplification is measured by the root mean square error, computed as the difference between the scalar field value at vertices of the original and the simplified meshes. The fourth dataset represents a molecule in solution, where the surface of the molecule is stored as the embedded structure. Error introduced by the simplification is measured as the rms distance between the two surfaces.

We simplify the four datasets and measure the root mean square error, the time taken and the standard deviation of dihedral, solid and face angles during simplification, see Table 7.1 for details. The average values of the three types of angles remain nearly constant at around 1.22, 0.53 and 1.05 radians.

**Topology violation without using Link Conditions:** The quadric error metric as described in Section 5.1 aims to preserve the geometry of the mesh and the embedded structure. We now illustrate, using an example, the importance of link conditions to ensure topology preservation.

Figure 7.1 shows a thin ring-like section in the embedded structure of the Liquid Oxygen Post dataset. The magnified view shows that the ring-like section is a 2-manifold with boundary. A simplification without checking link conditions of $\widetilde{K}$ creates a principal edge (i.e. no cofaces), thus violating the topology of the embedded structure.

(a) (b) (c)

Figure 7.1: Topology violation of embedded structure. (a) Embedded structure with a ring like portion. (b) Magnified view of the ring like portion which is a 2-manifold with boundary. (c) After simplification, a principal edge appears (shown in red) and violates topology.

(a) Bucky Ball

| % | #vert | rms | time | dihedral | solid | face |
|---|---|---|---|---|---|---|
| 100 | 331485 | 0 | 0 | 0.383 | 0.508 | 0.383 |
| 50 | 165742 | 0.016 | 44 | 0.397 | 0.463 | 0.361 |
| 30 | 99445 | 0.023 | 67 | 0.431 | 0.455 | 0.368 |
| 20 | 66297 | 0.029 | 80 | 0.458 | 0.456 | 0.377 |
| 10 | 33148 | 0.034 | 94 | 0.477 | 0.461 | 0.383 |
| 5 | 16574 | 0.039 | 103 | 0.489 | 0.468 | 0.388 |
| 2 | 6629 | 0.064 | 117 | 0.497 | 0.477 | 0.393 |

(b) Plasma64

| % | #vert | rms | time | dihedral | solid | face |
|---|---|---|---|---|---|---|
| 100 | 276446 | 0 | 0 | 0.338 | 0.495 | 0.343 |
| 50 | 138223 | 0.028 | 36 | 0.354 | 0.435 | 0.319 |
| 30 | 82933 | 0.029 | 54 | 0.393 | 0.423 | 0.327 |
| 20 | 55289 | 0.029 | 64 | 0.424 | 0.423 | 0.336 |
| 10 | 27644 | 0.028 | 74 | 0.442 | 0.427 | 0.343 |
| 5 | 13822 | 0.030 | 81 | 0.453 | 0.431 | 0.348 |
| 2 | 5528 | 0.032 | 87 | 0.460 | 0.437 | 0.351 |

(c) Liquid Oxygen Post

| % | #vert | rms | time | dihedral | solid | face |
|---|---|---|---|---|---|---|
| 100 | 126890 | 0 | 0 | 0.571 | 0.575 | 0.608 |
| 50 | 63445 | 0.039 | 20 | 0.607 | 0.602 | 0.612 |
| 30 | 38067 | 0.073 | 29 | 0.637 | 0.627 | 0.615 |
| 20 | 25378 | 0.089 | 33 | 0.661 | 0.646 | 0.617 |
| 10 | 12689 | 0.098 | 39 | 0.676 | 0.658 | 0.619 |
| 5 | 6344 | 0.122 | 42 | 0.686 | 0.667 | 0.622 |
| 2 | 2537 | 0.111 | 46 | 0.691 | 0.674 | 0.624 |

(d) Molecule

| % | #vert | rms | time | dihedral | solid | face |
|---|---|---|---|---|---|---|
| 100 | 34920 | 0 | 0 | 0.533 | 0.380 | 0.415 |
| 50 | 17460 | 0.008 | 9 | 0.607 | 0.495 | 0.449 |
| 30 | 10476 | 0.013 | 12 | 0.655 | 0.569 | 0.472 |
| 20 | 6984 | 0.021 | 14 | 0.690 | 0.619 | 0.490 |
| 10 | 3492 | 0.047 | 18 | 0.711 | 0.651 | 0.502 |
| 5 | 1746 | 0.127 | 22 | 0.722 | 0.666 | 0.509 |
| 2 | 698 | 0.279 | 26 | 0.725 | 0.671 | 0.511 |

Table 7.1: Results of simplification of three isosurface and a molecular surface dataset. The time taken is measured on a 2 GHz Intel Xeon CPU.

# Chapter 8

# Conclusion and Future Work

We prove theoretically the correctness of the technique proposed by [1, 2] for topology preserving simplification of meshes with embedded structures. Our approach results in a unified proof for 2D and 3D meshes. We also demonstrate usefulness of this technique in applications like isosurface topology preservation and molecular surface topology preservation. Besides preserving the scalar field and creating good quality mesh elements, our implementation also ensures that the geometry of the embedded structure in preserved.

In this report, we have analysed the necessity of the mesh simplification technique for 2D meshes with 1D embedded structures. The necessity of the technique in the case of 3D meshes needs to be analysed. In the current work, we do a detailed case analysis for computing order of simplices. We need to explore if there are other approaches to identify order of a simplex. Since such techniques would measure the topological complexity of simplices, they may provide better insights in identifying important features of a mesh.

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