Supplementary Material for "Analyzing Time-Varying Scalar Fields using Piecewise-Linear Morse-Cerf Theory"

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1 PL MORSE FUNCTIONS

Definition 1.1 (PL Manifold)**.** A topological manifold M is called a *PL (piecewise linear) manifold* if it is equipped with a covering $(M_i)_{i\in I}$ by charts such that all coordinate transition maps between overlapping charts are piecewise linear homeomorphisms between open subsets of Euclidean space.

A *triangulation* of a manifold $\mathbb{M}$ refers to a simplicial complex K such that the geometric realization $|K|$ is homeomorphic to $\mathbb{M}$. Any compact PL d-manifold admits a triangulation such that the link of every k-simplex is homeomorphic to a $(d-k-1)$-sphere. Such a triangulation is called a *combinatorial d-manifold*.

Given a triangulation, assigning values to the vertices determines a continuous PL function on $\mathbb{M}$ via linear interpolation over each simplex.

The *lower link* of a vertex u, with respect to a PL function f, is the subcomplex of the link $\mathrm{lk}(u)$ consisting of simplices all of whose vertices have function values less than or equal to $f(u)$. That is,

$$\mathrm{lk}^-(u) = \{\tau \in \mathrm{lk}(u) \mid \forall v \in \tau,\, f(v) \le f(u)\}.$$

PL Morse functions are piecewise linear analogues of smooth Morse functions on combinatorial manifolds. However, not all PL functions are PL Morse. To understand the structure of PL Morse functions, we first introduce a broader class of functions.

Definition 1.2 (Generic PL Function)**.** Let M be a combinatorial d-manifold. A function $f : M \to \mathbb{R}$ is called a *generic PL function* if:

- f is linear on each simplex of the triangulation;
- $f(v) \neq f(w)$ for any two distinct vertices v, w of M.

Given a generic PL function, one can define critical and regular points. In contrast to the smooth setting, a critical point in the PL setting can have a *multi-index*. In practice, we often encounter functions with multi-saddles for which generic PL functions provide a convenient framework.

We define regularity and criticality using the homology of the lower link of a generic PL function [2].

Definition 1.3 (Regular Point)**.** A vertex v with $f(v) = a$ is called *homologically regular* for f if

$$\dim_{\mathbb{F}} H_i(\mathrm{lk}^-(v);\mathbb{F}) = 0 \quad \text{for all } -1 \le i \le n-1,$$

for some (equivalently, any) field $\mathbb{F}$. We adopt the convention:

$$H_{-1}(\mathrm{lk}^-(v)) = \begin{cases} 0 & \text{if } \mathrm{lk}^-(v) \neq \emptyset, \\ \mathbb{F} & \text{if } \mathrm{lk}^-(v) = \emptyset. \end{cases}$$

Definition 1.4 (Critical Point and Multi-Index)**.** A vertex v with $f(v) = a$ is called *homologically critical* for f with *multi-index* $(k_0, k_1, \ldots, k_n)$ if

$$\dim_{\mathbb{F}} H_{i-1}(\mathrm{lk}^-(v);\mathbb{F}) = k_i \quad \text{for all } 0 \le i \le n.$$

A critical point v is said to be *non-degenerate* if its multi-index satisfies: exactly one $k_i = 1$, and all other $k_j = 0$ for $j \neq i$.

Definition 1.5 (PL Morse Function)**.** A generic PL function is called a *PL Morse function* if all of its critical points are non-degenerate.

Figure 1 shows an illustration of the type of critical points in a 2-manifold. The criticality is determined by the homology of the lower link.

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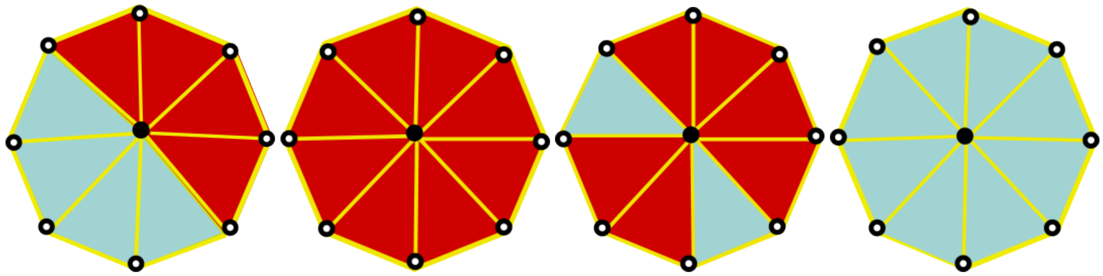

(a) Regular, $\beta = (0,0,0)$ (b) Minimum, $\beta = (1,0,0)$ (c) 1-Saddle, $\beta = (0,1,0)$ (d) Maximum, $\beta = (0,0,1)$

Figure 1: Types of critical points and their homological index β for a combinatorial 2-manifold. The lower star of the vertex (black dot) is shown in blue.

2 ECC AS SUM OF LOCAL ECC CONTRIBUTIONS

Let M be a combinatorial d–manifold and $f: M \to \mathbb{R}$ a generic PL function. A PL function on a combinatorial manifold with generic property is called "Morse" in Bestvina's [1] terminology; however, it is important to note that this terminology differs from the definition of a PL Morse function in contemporary literature which is more restrictive. For each vertex $v \in M_0$ and real number $s \in \mathbb{R}$ define the *local Euler Characteristic Curve* (local-ECC)

$$\chi_v(s) := \begin{cases} \sum_{i=0}^{d} (-1)^i \widetilde{\beta}_{i-1}\big(\mathrm{lk}^-(v)\big), & \text{if } f(v) \le s, \\ 0, & \text{otherwise.} \end{cases}$$

For $s \in \mathbb{R}$, define the sub-level set $M_s := f^{-1}\big((-\infty, s]\big)$.

The *Euler characteristic curve* (ECC) [3] is a topological descriptor of the scalar field f. It is defined as the following map

$$\mathscr{E}: \mathbb{R} \longrightarrow \mathbb{R}, \qquad \mathscr{E}(s) := \chi\big(M_s\big).$$

Theorem 2.1. *Let M be a combinatorial d–manifold and $f: M \to \mathbb{R}$ a generic PL function whose image lies in a compact interval $J \subset \mathbb{R}$. Then, for every $s \in \mathbb{R}$,*

$$\mathscr{E}(s) \;=\; \sum_{v \in M_0} \chi_v(s).$$

Proof. Label the vertices of M by $v_1, \ldots, v_N$ such that $f(v_1) < f(v_2) < \cdots < f(v_N)$.

Let $t_i = f(v_i)$. Given $s \in J$, let $R := \max\{\, i \mid t_i < s \,\}$. Choose numbers $s_0, s_1, \ldots, s_R \in \mathbb{R}$ satisfying

$$s_{i-1} < t_i < s_i \quad , \quad (s_{i-1}, s_i) \cap f(M_0) = \{t_i\} \quad (\text{ for } 1 \le i < R) \quad \text{and} \quad s_R := s \ge t_R.$$

For $1 \le i \le R$ define

$$A_i := f^{-1}(s_{i-1}), \qquad J_i := [s_{i-1}, s_i], \qquad B_i := f^{-1}(J_i).$$

(Notice $A_0 = \varnothing$ and $B_R = M_s$.)

By Proposition 2.7 of [1], there is a homotopy equivalence rel A_i,

$$(B_i, A_i) \;\simeq\; (A_i \cup C_i, A_i),$$

where C_i is the cone on $\mathrm{lk}^- v_i$ (with cone point v_i and base contained in A_i).

We will denote cone of X by $\mathrm{Cone}(X)$ and suspension of X by ΣX. For each $k \in \mathbb{N}_0$,

$$\begin{aligned} H_k\big(A_i \cup C_i, A_i\big) &\cong \widetilde{H}_k\big((A_i \cup C_i)/A_i\big) && \text{(excision for a good pair)} \\ &\cong \widetilde{H}_k\big(\Sigma\, \mathrm{lk}^-(v_i)\big) && \text{since } \mathrm{Cone}(X)/X \cong \Sigma X \\ &\cong \widetilde{H}_{k-1}\big(\mathrm{lk}^-(v_i)\big) && \text{because } \widetilde{H}_k(\Sigma X) \cong \widetilde{H}_{k-1}(X). \end{aligned}$$

Therefore,

$$H_k(B_i, A_i) \;\cong\; \widetilde{H}_{k-1}\big(\mathrm{lk}^-(v_i)\big) \quad \text{for all } k \ge 0,$$

so that

$$\chi(B_i, A_i) = \sum_{k=0}^{d} (-1)^k \widetilde{\beta}_{k-1}\big(\mathrm{lk}^-(v_i)\big) = \chi_{v_i}(t_i).$$

For each CW pair (X,A), the Euler characteristic is additive: $\chi(X) = \chi(A) + \chi(X,A)$.

Therefore, by expressing $\chi(M_s)$ as a sum of the relative Euler characteristics $\chi(B_i,A_i)$, we obtain

$$\mathscr{E}(s) = \chi(M_s) = \sum_{i=1}^{R} \chi(B_i,A_i) = \sum_{i=1}^{R} \chi_{v_i}(t_i) = \sum_{v \in M_0} \chi_v(s).$$

□

3 Computing the Cerf diagram

Algorithm 1 computes the Cerf diagram of a PL time-varying scalar field defined on a simplicial complex K. The time-varying scalar field f_t is available as a sample at T time steps and the function value at a vertex is specified at each time step, $\{f_t(v)\}, t \in \{0,\dots,T-1\}, v \in V$. For any $t-1 < t' < t$, $f_{t'}(v)$ is assumed to be linearly interpolated between $f_{t-1}(v)$ and $f_t(v)$, resulting in a time-varying scalar field over a continuous time interval where the function value at each vertex is a PL function of time. The vertex curves in the vertex diagram are therefore PL curves, and their crossings can be computed efficiently. Further, since criticality is determined by the lower link, we only need to examine crossings between pairs of vertices that lie in the link of each other. We assume that each f_t is generic for $t \in \{0,\dots,T-1\}$. We further assume that the family $\{f_t\}$ is generic. For a positive integer m, $[m]$ denotes the set $\{0,\dots,m\}$.

Algorithm 1: Compute Cerf Diagram

Input : K, T(#timesteps), $\{f_t(v)\}, t \in [T-1], v \in V$
Output: Set of line segments C in the Cerf diagram of $\{f_t\}$

Procedure COMPUTE-CERF-DIAGRAM$(K, \{f_t\})$
- Compute $S_v \leftarrow \{((t, f_t(v)), (t+1, f_{t+1}(v))) \mid t \in [T-2], \forall v \in V\}$ `//` S_v `stores the graph of` v `as a collection of line segments represented by their end points.`
- Compute all intersections between S_u, S_v, where u, v are in the link of each other. Let $\mathscr{I}_v \leftarrow \{(t,u,v,f_t(v)) \mid k < t < k+1, k \in [T-2], f_t(u) = f_t(v), f_k(u) > f_{k+1}(v), u \in lk(v)\}$ `//` $\mathscr{I}_v$ `records all instances when a vertex` u `in the link of` v `crosses over and enters the lower link of` v.
- $\mathscr{I} = \cup_{v \in V} \mathscr{I}_v \cup \{(t,v,f_k(v)) \mid t \in [T-1]\}$ `// Tuples of the form` $(t,v,f_k(v))$ `are required as the graph of` v `changes at time steps` $t \in [T-1]$
- Sort $\mathscr{I}$ with respect to t
- $lk^{(0)-}(v) \leftarrow \{u \in V(lk(v)), f_0(u) < f_0(v)\}$ `// Tracks vertices in` $lk^-(v)$
- **for** $(t,u,v,f_t(v)) \in \mathscr{I}, t \notin [T-1]$ **do**
  - $lk^{(0)-}(v) \leftarrow lk^{(0)-}(v) \cup \{u\}$ `//` u `enters` $lk^-(v)$
  - $lk^{(0)-}(u) \leftarrow lk^{(0)-}(u) \backslash \{v\}$ `//` v `exits` $lk^-(u)$
  - Update $lk^-(v), lk^-(u)$
  - **for** $z \in \{u,v\}$ **do**
    - **if** $\widetilde{H}_*(lk^-(z)) \neq 0$ *and next tuple in* $\mathscr{I}$ *containing* z *occurs at* t' **then**
      - add $(t, f_t(z), t', f_{t'}(z), z, \beta(z))$ to C

We note that the algorithm does not require the time-varying family to be PL Morse, since the Cerf diagram records the entire homological index of each critical vertex, thereby also taking into account degenerate critical points.

We assume that the size of the link of a vertex is upper bounded by l. Between two successive time steps, the number of intersections encountered while processing a vertex is at most $O(l)$. Hence, all intersections are identified in $O(nl)$ time. Each intersection triggers a computation of homology of the lower link, which requires $O(l^{\omega})$ time. Here, ω is the exponent used to express the time complexity of matrix multiplication. So, the total runtime is $O(Tnl^{\omega+1})$, where T is the number of time steps.

4 TV-ECC: Stability and Computation

For generic PL functions f and g, we define

$$\Delta(\{f\},\{g\}) := \int_{\mathbb{R}} \left| \mathscr{E}_f(s) - \mathscr{E}_g(s) \right| \mathrm{d}s.$$

Theorem 4.1. *For generic PL functions, Δ is stable with respect to the L^{∞} norm.*

Proof. Let f, g be two generic PL functions on K. Consider the one–parameter family

$$F : K \times [0,1] \longrightarrow \mathbb{R}, \qquad F(x,t) = (1-t) f(x) + t\, g(x).$$

Assume F is generic; otherwise perturb it arbitrarily slightly so that it is. Let $0 < t_1 < \cdots < t_m < 1$ be the instants at which F fails to be generic, i.e. some pair of vertices share the same function value. Fix one such instant t_k for which $F(v,t_k) = F(w,t_k)$.

Continuity of $\mathscr{E}_{F_t}$ at t_k. Take $t < t_k < t'$ sufficiently close to t_k. Write

$$\mathscr{E}_{F_t} = \sum_{x \in V(K)} \mathbf{1}_{\{F_t(x) \leq s\}}\, \chi_x(t).$$

Between t and t' the summands χ_x remain unchanged for $x \notin \{v, w\}$, hence

$$\mathscr{E}_{F_t} - \mathscr{E}_{F_{t'}} = \sum_{x \notin \{v,w\}} \big(\mathbf{1}_{\{F_t(x) \leq s\}} - \mathbf{1}_{\{F_{t'}(x) \leq s\}}\big)\chi_x(t) + \big(\mathbf{1}_{\{F_t(v) \leq s\}}\chi_v(t) + \mathbf{1}_{\{F_t(w) \leq s\}}\chi_w(t)\big) - \big(\mathbf{1}_{\{F_{t'}(v) \leq s\}}\chi_v(t') + \mathbf{1}_{\{F_{t'}(w) \leq s\}}\chi_w(t')\big).$$

Let

$$a \;=\; \lim_{|t-t'| \to 0} F_t(v) = F_t(w) = F_{t'}(v) = F_{t'}(w).$$

Using Theorem 3.4 of the main paper, it follows that $\mathscr{E}_{F_t} - \mathscr{E}_{F_{t'}} \longrightarrow 0$ as $t \to t'$. Thus $\mathscr{E}_{F_t}$ extends continuously to $[0,1]$.

Assume (without loss of generality) that $f(v) > g(v)$. Then

$$\big|\mathbf{1}_{\{F_{t_{i+1}}(x) \leq s\}}\chi_v(t_{i+1}) - \mathbf{1}_{\{F_{t_i}(x) \leq s\}}\chi_v(t_i)\big| \;\leq\; \mathbf{1}_{\{F_{t_{i+1}}(x) \leq s < F_{t_i}(x)\}}\, l,$$

where l is an upper bound on the number of simplices in the link of a vertex.

Therefore

$$|\mathscr{E}_f - \mathscr{E}_g| \;\leq\; \sum_{i=0}^{m} \big|\mathscr{E}_{F_{t_{i+1}}} - \mathscr{E}_{F_{t_i}}\big| \;\leq\; \sum_{\substack{x \\ f(x) > g(x)}} \mathbf{1}_{\{g(x) \leq s < f(x)\}}\, l \;+\; \sum_{\substack{x \\ g(x) > f(x)}} \mathbf{1}_{\{f(x) \leq s < g(x)\}}\, l.$$

Hence, under the L_1 norm,

$$|\mathscr{E}_f - \mathscr{E}_g| \;\leq\; n\, l\, \|f - g\|_\infty.$$

□

As a consequence we obtain the following stability result for the following distance measure between time–varying scalar fields.

$$\Delta\big(\{f_t\}, \{g_t\}\big) \;:=\; \int_0^1 \int_{\mathbb{R}} \big|\mathscr{E}_{f_t}(s) - \mathscr{E}_{g_t}(s)\big|\, \mathrm{d}s\, \mathrm{d}t.$$

Theorem 4.2. *Let $\{f_t\}, \{g_t\}$ be two generic PL families on a complex K with n vertices and with upper bound l on the size of the link of each vertex. Then*

$$\Delta\big(\{f_t\}, \{g_t\}\big) \;\leq\; n\, l \int_0^1 \|f_t - g_t\|_\infty\, \mathrm{d}t.$$

Computation of TV-ECC. For two generic PL functions f, g, $\Delta(f,g) = \int_{\mathbb{R}} \big|\mathscr{E}_f(s) - \mathscr{E}_g(s)\big|\, \mathrm{d}s$ can be computed exactly for $\mathscr{E}_f, \mathscr{E}_g$, and therefore $|\mathscr{E}_f - \mathscr{E}_g|$, are piecewise constant functions. If c is an upper bound on the number of critical points of f, g then $\Delta(f,g)$ can be computed in $O(c)$ time. For two generic families $\{f_t\}, \{g_t\}, t \in I$ the outer integral of $\Delta(\{f_t\}, \{g_t\})$ is approximated using Monte-Carlo integration in $O(mc)$ time, where m is the number of samples taken.

References

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